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WEAKLY NONLINEAR INTERACTIONS AND WAVE-TRAPPING.(U)

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ABSTRACT

When the flow over a submerged, round, upright cylinder, situated in a large ocean, is forced by a train of plane waves, linear theory (Yamamuro, 1981) shows that the response can be abnormally large for certain forcing frequencies. The aim of this paper is to present a weakly nonlinear theory, where wave interactions, arising from the quadratic terms in the free-surface boundary conditions, can yield abnormally large responses.

A specific interaction will be considered between a flow at a subharmonic frequency and a flow at the driving frequency. The reason for considering such an interaction derived from a consideration of some experimental results of Barnard, Pritchard and Provis (1981)

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Key words: surface gravity water waves, trapping modes, nonlinear interactions.

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SIGNIFICANCE AND EXPLANATION

Observations of unusually large response in the wave records at Macquarie Island, in the ocean south of Australia, have been explained as being due to the excitation of nearly resonant trapped modes. This phenomenon has motivated much experimental investigation of trapped modes around a submerged upright cylinder in water of constant depth. Theories, based on the small-amplitude approximation, lead to predictions of resonances. However, some experimental work indicated the presence of subharmonic components in the wavefield. This paper develops a possible explanation for this by showing that nonlinear interactions could yield resonance of subharmonic modes. Moreover, a particular set of conditions is presented, in which the amplitudes of plane waves incident on the obstacle are significantly magnified above it by subharmonic resonance.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

WEAKLY NONLINEAR INTERACTIONS AND WAVE-TRAPPING

Yuriko Renardy

§1. Introduction.

If a train of plane waves of a certain frequency is incident on a submerged round sill, situated in a large ocean (Figure 1), linear theory (Yamamuro, 1981) predicts that the overall amplitudes over the sill may become much larger than those of the deeper ocean. Such a phenomenon may be called "wave-trapping" or "near-resonance", referring to the unusually large response. The purpose of this paper is to examine a contribution of nonlinear effects to the wave-trapping phenomenon.

Let the fluid over the submerged sill be denoted by D_1 and the fluid outside this region by D_2. The total domain for the flow will be denoted by $D = D_1 \cup D_2$. The boundary conditions at the free surface are nonlinear. However, the amplitudes of the motion in D_2 are assumed to be small enough so that those conditions may be linearized. An investigation is made of a wave-trapping phenomenon in which 'small-amplitude' waves enter D_1 and become magnified, so that in this region some nonlinear terms are included in the free-surface conditions. A possible mechanism of magnification is a wave interaction arising from the quadratic terms in the free-surface boundary conditions (Phillips, §3.8), and it seems appropriate to restrict attention to these in an initial investigation. Therefore, if σ is the driving frequency, attention will be limited to interactions, in D_1, with waves of frequencies $\sigma/2$ and 2σ.

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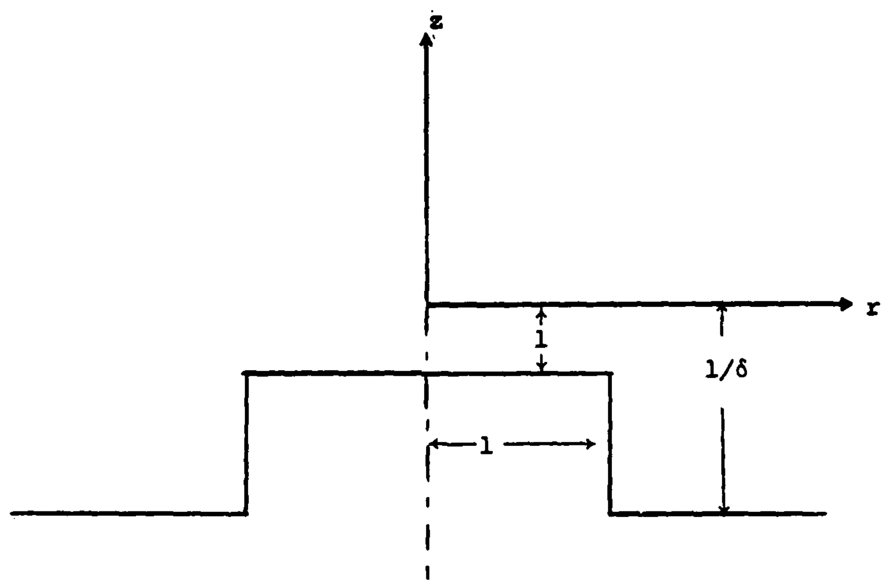

Figure 1

The structure of the paper is as follows.

§2, §3. In §2, the equations governing the flow are presented. An attempt to satisfy them by a modal decomposition of the velocity potential in the radial variable leads to an infinite matrix equation. The organization of the solution method for this set of equations is described in §3. A linear calculation (Yamamuro, 1981) shows that, in order to make the velocity continuous across the sill-edge, the wave-field necessarily contains modes that decay away from the sill-edge and are named "decaying modes". Resonance of such modes will not be considered here. Hence the solutions exhibiting large response are assumed to be combinations of eigenstates, each of which consists of a principal part, which is wavelike, together with associated decaying modes.

§4. In order to calculate the amplitudes of the wave motion, it is necessary to solve a 'homogeneous' problem in D, namely a linearized problem with no incident waves. This is investigated in §4. The problem posed in D_2 is solved from an existing linear theory. The flow structure in D_2 is then used to generate a boundary condition at the sill-edge $r = 1$ so that the problem is reduced to a boundary-value problem in D_1. It is found from a separation of variables that an eigenstate in D_1, periodic in time with a complex-valued frequency Ω, consists of a 'wavelike' mode and an infinite number of 'decaying' modes. In addition, there are an infinite number of complex-valued coefficients to be determined from the boundary conditions.

The eigenfrequencies Ω and the coefficients have been determined simultaneously by finding the zeros of an expression in the depth variable. A collocation method is used and consists of applying that expression at N values of the depth variable and including the first N modes of each eigenstate. This yields an $N \times N$ matrix, whose zeros can be found by the following method. The condition number of the matrix is computed over a grid of complex frequencies and those with relatively large condition numbers are taken to be approximations to the Ω's. This method was, however, time-consuming and an alternative, non-standard, method was devised. The method is an iterative scheme, based on the smallness of the response of the decaying modes compared with that of the wavelike modes.

§5. The solution to the homogeneous problem provides a set of orthogonal functions, a combination of which can be used to form an expression for the surface elevation. The total sill solution is then written as a suitable combination of the orthogonal functions, superposed on the linear solution. In §5, an example of such a nonlinear interaction is constructed. The example chosen here displays subharmonic resonance and was motivated by previous work

on edge waves (Guza & Davis, 1974; Minzoni & Whitham, 1977; Rockliff, 1978). The interaction involves three modes: two $\cos 2\theta$ modes at the forcing frequency, and a $\cos\theta$ mode at half that frequency. It is found that near-resonance occurs for two ranges of forcing frequencies. One range occurs near an eigenfrequency of the $\cos 2\theta$ mode, and the other occurs near twice an eigenfrequency of the $\cos\theta$ mode.

§2. Formulation of the problem.

It is assumed that the motion is inviscid and irrotational. The domain of the flow above the sill $(r < 1)$ is denoted by D_1 and that outside the sill region $(r > 1)$ by D_2. The amplitudes of the flow in D_2 are assumed to be sufficiently small so that the free-surface boundary conditions may be linearized and applied at an equilibrium level. Such a linearization is not assumed in D_1. The geometry of the problem is shown in Figure 1. Cylindrical coordinates r, θ, z are used: r is measured outward in units of the sill radius a, and z is measured vertically upward in units of the undisturbed water depth d above the sill. The depth outside the sill is denoted by $D = d/\delta$. The velocity potential $\phi(r,\theta,z,t)$, which is in units of $d^2\sigma$, satisfies Laplace's equation, namely,

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \frac{a^2}{d^2}\phi_{zz} = 0 \, . \tag{2.1}$$

Let $\eta(r,\theta,t)$ denote the surface displacement and x denote the horizontal coordinate in the direction $\theta = 0$. A train of plane waves is assumed to be incident from the positive x-axis and is represented by the real part of

$$\eta = |\eta|_I \, e^{-i(kx + \sigma t)} \tag{2.2}$$

where k is the positive real root of

$$(KD/a) \tanh (KD/a) = D\sigma^2/g. \tag{2.3}$$

The free-surface boundary conditions are:

at $z = \eta(r,\theta,t)$, $0 < \theta < 2\pi$, $r > 0$, a kinematic condition

$$\phi_z = \dot{\eta}/\sigma + (d^2/a^2)(\nabla_h \phi) \cdot (\nabla_h \eta) \tag{2.4}$$

where $\nabla_h \equiv (\partial/\partial x), \partial/\partial y)$, and a dynamical condition

$$d^2\sigma(\phi_{tt} + (g/d)\phi_z) + \frac{\partial}{\partial t}(\underline{u}^* \cdot \underline{u}^*) + \tfrac{1}{2}(\underline{u}^* \cdot \nabla(\underline{u}^* \cdot \underline{u}^*)) = 0 \tag{2.5}$$

where

$$\underline{u}^* = d^2\sigma(\phi_r/a, \, \phi_\theta/ar, \, \phi_z/d). \tag{2.6}$$

The small-amplitude approximation of condition (2.4) to second order is found (Phillips, §3.1) to be, for $z = 0$, $0 \leq \theta \leq 2\pi$, $r \geq 0$,

$$\phi_{tt} + (g/d)\phi_z = -\eta \frac{\partial}{\partial z} (\phi_{tt} + (g/d)\phi_z) - \frac{\partial}{\partial t}(\underline{u}^* \cdot \underline{u}^*)/d^2\sigma. \tag{2.7}$$

The possibility of a resonance, in which the right-hand-side becomes of the same order as the combined expression on the left-hand-side, will be investigated for the sill region. The possibility of a similar kind of resonance through condition (2.5) will not be considered. Under these assumptions, the free-surface boundary conditions to be used are: for $z = 0$,

$$\phi_{tt} + (g/d)\phi_z = \begin{cases} -\eta \frac{\partial}{\partial z} (\phi_{tt} + (g/d)\phi_z) - \frac{\partial}{\partial t}(\underline{u}^* \cdot \underline{u}^*)/d^2\sigma & \text{for } D_1 \\ 0 & \text{for } D_2 \end{cases} \tag{2.8}$$

and

$$\eta_t = \sigma\phi_z \qquad \text{for } D . \tag{2.9}$$

As mentioned in §1, only flows of frequencies σ, 2σ and $\sigma/2$ will be investigated, so that condition (2.8) can be expressed in the following form:

$$\phi_{tt} + g/d\ \phi_z = \begin{cases} f_1(r,\theta)\ e^{-i\sigma t} + f_2(r,\theta)e^{-\frac{1}{2}i\sigma t} + f_3(r,\theta)e^{-2i\sigma t} + * & \text{for } D_1 \\ 0 & \text{for } D_2 . \end{cases} \tag{2.10}$$

Here, the asterisk denotes the complex conjugate of preceding terms. The f_i $(i = 1,2,3)$ are complex functions of ϕ and are determined. Once ϕ is known and substituted into equations (2.8) and (2.9).

The boundary condition on rigid surfaces is one of zero normal velocity, i.e.

$$\phi_z = 0 \quad \text{for} \quad z = -1,\ r < 1 \quad \text{and} \quad z = -1/\delta,\ r > 1 , \tag{2.11}$$

and

$$\phi_r = 0 \qquad \text{for} \quad r = 1,\ -1/\delta < z < -1. \tag{2.12}$$

At the sill-edge, the velocity is assumed to be continuous, a condition that is equivalent to the ϕ and ϕ_r being continuous there. At large distances from the sill, the wavefield is assumed to consist of the incident wave, and waves that either decay or radiate outward.

§3. Form of 'resonant' solutions.

The problem formulated in §2, with (2.8) and (2.9) as the free-surface conditions, is essentially a superposition of two problems. Hence the solutions are expressed as $\phi = \phi_L + \phi_N$. The ϕ_L satisfies the linearized free-surface conditions, namely,

$$\phi_{L\,tt} + (g/d)\phi_{Lz} = 0 \quad \text{at} \quad z = 0 \,, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r < \infty \,, \tag{3.1}$$

and is forced by the plane waves represented by (2.2) and (2.3). The ϕ_N is constructed to make the total velocity potential ϕ satisfy the nonlinear free-surface conditions (2.8) and (2.9), and is not forced by the plane waves. Both ϕ_L and ϕ_N satisfy the conditions (2.11) and (2.12) at the solid walls. The ϕ_N interacts with ϕ_L through the free-surface conditions and the solutions of interest are those in which ϕ_N becomes comparable with ϕ_L.

The linear solution ϕ_L has been calculated (Yamamuro, 1981) by a separation of variables for the regions $0 \leq r < 1$ and $r > 1$, and the velocity is made continuous throughout the flow.

The ϕ_N are expressed as a linear combination of orthogonal functions that satisfy equation (2.1) and conditions (2.8), (2.11), (2.12), and the radiation condition that the flow be non-growing at large r. Let the part of ϕ_N in D_1 be denoted by ϕ_{N1} and that in D_2 by ϕ_{N2}. ϕ_{N2} is to satisfy linearized conditions and to consist of flows of frequencies σ, $\sigma/2$ and 2σ. Separation of variables then yields

$$\phi_{N2} = e^{-i\sigma t}\sum_{m=0}^{\infty} \cos m\theta \{B_{m0}H_m^{(1)}(a\lambda_0 r/D)\cosh\lambda_0(\delta z+1)$$

$$+ \sum_{n=1}^{\infty} B_{mn}K_m(a\lambda_n r/D)\cos\lambda_n(\delta z+1)/K_m(a\lambda_n/D)\} + * \tag{3.2}$$

+ {similar expressions for flows at frequencies $\sigma/2$ and 2σ}

where $\{\pm\lambda_0, \pm i\lambda_1, \pm i\lambda_2, \cdots\}$ are the roots of the dispersion relation (Davis & Hood, 1976)

$$\lambda \tanh \lambda = D\sigma^2/g. \tag{3.3}$$

Notations for the Bessel functions $H_n^{(1)}(x)$ and $K_m(x)$ are those used by Abramowitz and Stegun (1972). The complex coefficients B_{mn} are yet to be determined. However a boundary condition for ϕ_{N1} is first constructed by the elimination of the B_{mn}.

At $r = 1$, the condition that ϕ_N and $\partial\phi_N/\partial r$ be continuous is:

$$\partial\phi_{N2}/\partial r = \begin{cases} \partial\phi_{N1}/\partial r & \text{for } -1 < z < 0 \\ 0 & \text{for } -1/\delta < z < -1, \end{cases} \tag{3.4}$$

and

$$\phi_{N2} = \phi_{N1} \quad \text{for} \quad -1 < z < 0 \tag{3.5}$$

The expression (3.2) is substituted in equation (3.4). In what follows, the discussion is focused on flow at one of the three frequencies (σ, $\sigma/2$ or 2σ) denoted by ω. The orthogonality of the set $\{\cosh \lambda_0(\delta z+1),$ $\cos \lambda_n(\delta z+1): n = 1,2,\cdots\}$ over $-1/\delta < z < 0$ is used and integration over z yields the B_{mn} in terms of $\partial\phi_{N1}/\partial r$ at $r = 1$. These equations are then used in equation (3.5) to eliminate the B_{mn}. A boundary condition results for $\phi_m(z)$, the coefficient of $\cos m\theta e^{-i\omega t}$ in ϕ_{N1} at $r = 1$:

$$\phi_m(z) = \int_{-1}^{0} \frac{\partial\phi_m}{\partial r}(z')k_\omega(z,z')dz' \quad \text{for} \quad -1 < z < 0 , \tag{3.6}$$

where

$$K_\omega(z,z') = \frac{H_m(a\lambda/D) \cosh \lambda_0(\delta z+1) \cosh \lambda_0(\delta z'+1)}{H_m'(a\lambda_0/D)(a\lambda_0/D)\, h(\lambda_0)} + \sum_{n=1}^{\infty} \frac{K_m(a\lambda_n/D) \cos \lambda_n(\delta z+1) \cos \lambda_n(\delta z'+1)}{K_m'(a\lambda_n/D)(a\lambda_n/D)\, h(i\lambda_n)}. \tag{3.7}$$

Thus, the remaining problem is to solve for ϕ_{N1} using equations (2.1), (2.11) and (3.6). The details are given in §4. For each of the frequencies w (σ, $\sigma/2$ or 2σ), the corresponding flow in ϕ_{N1} is constructed as follows. The spatial variation of ϕ_{N1} is constructed to be that of the solutions $\Phi_{mn}(r,z)\cos m\theta e^{-i\Omega_{mn}t} + *$ which satisfy the "homogeneous" problem for ϕ_{N1}, i.e. condition (2.8) is replaced by the linearized form. The set $\{\Phi_{mn}(r,0)$: m fixed, $n = 1,2,\cdots\}$ can be shown to be orthogonal for $0 \leq r \leq 1$ by the application of Green's theorem (Jeffries & Jeffries, §5.081) to the domain D_1. Thus ϕ_{N1} can be written as

$$\phi_{N1} = \sum_{m=0}^{\infty} \cos m\theta \sum_{n=1}^{\infty} \{\alpha_{mn} e^{-i\sigma t/2} \Phi_{\alpha mn}(r,z)$$

$$+ B_{mn} e^{-i\sigma t}\Phi_{\beta mn}(r,z) + \gamma_{mn} e^{-2i\sigma t}\Phi_{\gamma mn}(r,z)\} + * \tag{3.8}$$

where $\Phi_{\alpha mn}(r,z)\cos m\theta\, e^{-i\Omega_{\alpha mn}t}$, $\Phi_{\beta mn}(r,z)\cos m\theta e^{-i\Omega_{\beta mn}t}$ and

$\Phi_{\gamma mn}(r,z)\cos m\theta\, e^{-i\Omega_{\gamma mn}t}$ are the solutions to the homogeneous problem for ω equal to $\sigma/2$, σ and 2σ respectively. The complex coefficients α_{mn}, β_{mn} and γ_{mn} are determined by condition (2.8), in which the orthogonality property of those functions is used. For example, the α_{mn} are determined by

$$\alpha_{mn}(-\sigma^2/4 + \Omega^2_{\alpha mn}) \int_0^1 \Phi^2_{\alpha mn}(r,0)rdr$$

$$= \int_0^1 \Phi_{\alpha mn}(r,0) \left[\frac{\int_0^{2\pi} f_2(r,0)\cos m\theta\, d\theta}{\int_0^{2\pi}\cos^2 m\theta\, d\theta}\right] rdr. \tag{3.9}$$

Similar equations hold for β_{mn} and γ_{mn}. These equations in which ϕ_L and ϕ_N interact will be referred to as 'interaction' equations.

It can be seen that the 'response' $|\alpha_{mn}|$ is inversely proportional to $|-\sigma^2/4 + \Omega^2_{\alpha mn}|$ so that in actual computations involving a particular geometry, only the one α_{mn}, whose $\Omega_{\alpha mn}$ is the closest to $\sigma/2$, is expected to be important. Similarly, at most one β_{mn} and one γ_{mn} are expected to be significant enough for inclusion in the interaction calculations.

We note in passing that the Longuet-Higgins eigenfunctions, referred to as 'free modes' in his paper (1967), cannot be used here for the construction of ϕ_{N1} because then the range of integration on the left hand side of (3.9) would extend to infinity and the integrand would grow almost exponentially with r.

§4. The homogeneous problem on D_1.

This problem consists of equation (2.1), with the boundary conditions:

$$\text{at } z = 0, \ \phi_{tt} + (g/d)\phi_z = 0 \tag{4.1}$$

$$\text{at } z = -1, \ \phi_z = 0 \tag{4.2}$$

at $r = 1$, condition (3.6) holds and w is assumed to be the frequency of the flow in D_2.

Separation of variables yields solutions of the form $\Phi(r,z)\cos m\theta\, e^{-i\Omega t}$, where

$$\Phi(r,z) = A_m J_m(akr/d)\cosh k(z+1) + \sum_{n=1}^{\infty} A_{mn}\, I_m(ak_n r/d)\cos k_n(z+1)/I_m(ak_n/d) \tag{4.3}$$

and A_m, A_{mn}, k, $k_{n,\Omega}$ are to be determined. Equations (4.1) and (4.2) yield a dispersion relation between k, k_n and Ω, namely

$$\left\{ \begin{matrix} k \\ ik_n \end{matrix} \right\} \tanh \left\{ \begin{matrix} k \\ ik_n \end{matrix} \right\} = d\Omega^2/g. \tag{4.4}$$

The A_m, A_{mn}, k and k_n are related through condition (3.6), which yields the following equation:

$$A_m L(k,z) + \sum_{n=1}^{\infty} A_{mn} L_n(k_n,z) = 0 \quad \text{for} \quad -1 < z < 0 \tag{4.5}$$

where the operators $L(k,z)$ and $L_n(k_n,z)$ are defined in the appendix. The wave numbers λ and λ_n, appearing in the operators L and L_n, correspond to the fixed frequency w of the flow in D_2. They are known through the dispersion relation

$$\left\{ \begin{matrix} \lambda \\ i\lambda_n \end{matrix} \right\} \tanh \left\{ \begin{matrix} \lambda \\ i\lambda_n \end{matrix} \right\} = Dw^2/g \tag{4.6}$$

where $\omega = \sigma$, $\sigma/2$ or 2σ. A matrix equation that expresses the A_{mn} in terms of A_n and also yields the Ω, is constructed by satisfying equation (4.5) at N values of z and neglecting the coefficients A_{mn} for $n > N$. Convergence with N was checked numerically. One way of obtaining the Ω's is to search through the complex plane by computing the condition number of the matrix over a grid in the complex plane. If a grid point yields a near-singular matrix, then that complex number can be taken to approximate an Ω. An alternative, less time-consuming method is now described.

§4.1. Iterative scheme.

The scheme is constructed to take advantage of the largeness of $|A_m|$, representing the response of the wavelike modes, as compared to $|A_{mn}|$ representing the response of the decaying modes, and of the property that $|A_{mn}|$ is smaller for larger n : i.e. the wavefield contains little of the modes that decay very fast away from the sill-edge. Equation (4.5) is multiplied by each of the functions in the set $\{\cosh k(z+1), \cos k_n(z+1):$ $n = 1,2,\cdots\}$ and integrated over $-1 < z < 0$. Since the elements of that set are orthogonal to each other, the resulting equations take the form:

for $m = 0,1,2,\cdots$

$$A_m X(k,\lambda,\lambda_n) + \sum_{s=1}^{\infty} A_{ms} X_s(k_s,\lambda,\lambda_n) = 0 \tag{4.7}$$

and for $p = 1,2,\cdots$

$$A_{mp} Y(k_p,k_p,\lambda,\lambda_n) + \sum_{\substack{s=1 \\ s\neq p}}^{\infty} A_{ms} Y(k_s,k_p,\lambda,\lambda_n) = -A_m Y(k,k_p,\lambda,\lambda_n). \tag{4.8}$$

The functions X, X_p and Y are defined in the appendix.

The iteration proceeds as follows. The first iterate for k, denoted by $k^{(0)}$, is calculated from the following reduced form of equation (4.7), in which all decaying modes are neglected:

$$J_m(ak/d)H_m'(a\lambda/D)f(k)\delta h(\lambda)\lambda - k\ J_m'(ak/d)H_m(a\lambda/D)[g(\lambda,k)]^2 = 0 \qquad (4.9)$$

where λ is known. The corresponding eigenfrequency $\Omega^{(0)}$ is calculated from

$$k^{(0)}\tanh k^{(0)} = d\Omega^{(0)2}/g, \qquad (4.10)$$

after which the $k_n^{(0)}$'s are calculated from equation (4.4). Next, the equations (4.8) are used to express A_{mn}/A_m for $n = 1,2,\cdots$. Then, on using these relations to eliminate the A_{mn}'s, equation (4.7) takes the form:

$$x_1(k) + \delta\lambda h(\lambda)x_2(k,k_n)H_m'(a\lambda/D) = 0 \qquad (4.11)$$

where $x_1(k)$ represents the left hand side of equation (4.9) and contains no decaying modes, $x_2(k,k_n)$ involves the decaying modes, and the notation is defined in the appendix.

The nth iterate $k^{(n)}$ $(n = 1,2,\cdots)$ is calculated by a Newton's method from equation (4.11) in which the term $x_2(k,k_n)$ is calculated at the known (n-1)th iterate, i.e.,

$$x_1(k^{(n)}) + \delta\ \lambda h(\lambda)\ x_2(k^{(n-1)},k_m^{(n-1)})H_m'(a\lambda/D) = 0\ . \qquad (4.12)$$

The corresponding $\Omega^{(n)}$ and $k_m^{(n)}$ are then calculated, as for the zeroth-iterate from equation (4.4).

A numerical check of the scheme was performed as follows. The Ω's were compared with the eigenfrequencies for the domain D denoted by Ω_D because they were expected to have similar values. The Ω_D's generalize the frequencies of the 'free modes' investigated by Longuet-Higgins (1967) using shallow-water theory, and were calculated in a similar way to the Ω's. The only difference was that in equation (4.11), λ was also an unknown. Therefore, at each step of the iteration, the two equations, (4.11) and

$$k \tanh k - \delta\ \lambda \tanh \lambda = 0\ , \qquad (4.13)$$

were solved simultaneously for k and λ by a Newton's method.

§5. Example of a near-resonance.

A particular set of conditions in which the foregoing theory yields near-resonance will be presented. In order to simplify computations, the parameters δ and d/a will be chosen to be small. The smallness of d/a ensures the smallness of the effect of the decaying modes in the flow in D_1 so that in the 'interaction' equations, such as (3.9), the decaying modes will be assumed to be negligible. However, the decaying modes will not be neglected in the computation of the Ω's since these are required to a high order of accuracy. Furthermore, the smallness of δ ensures that some of the Ω's will have very small imaginary parts, so that if the flow is forced near such an eigenfrequency, near-resonance is possible.

The experimental scales of Barnard, Pritchard and Provis (1981) were examined for the presence of near-resonant nonlinear interactions but were found not to yield them. However, a choice of scales which do are:

$$d = 2_{cm},\ d/a = .005,\ \delta = .002,\ |\eta|_I = .01.$$

In this case, the maximum amplitudes of the decaying modes is at least an order of magnitude less than that of the wavelike mode. Although these scales are unusual, there may be other realistic combinations where the nonlinear theory is applicable.

An interaction of three modes will be considered: the $(\cos\theta e^{-i\sigma t/2} + *)$ and $(\cos 2\theta e^{-i\sigma t} + *)$ modes in ϕ_N and the $(\cos 2\theta e^{-i\sigma t} + *)$ mode in ϕ_L. In the sill region, these are represented as follows:

$$\phi_L = A_2 J_2(akr/d) \cos 2\theta \cosh k(z+1) e^{-i\sigma t} + * + \text{decaying modes} \qquad (5.2)$$

$$\begin{aligned}\phi_N = {} & \alpha \cos\theta\, e^{-i\sigma t/2} J_1(a\nu r/d) \cosh \nu(z+1) + * + \text{decaying modes} \\ & + B \cos 2\theta\, e^{-i\sigma t} J_2(a\mu r/d) \cosh \mu(z+1) + * + \text{decaying modes}\end{aligned} \qquad (5.3)$$

where

$$k \tanh k = d\sigma^2/g, \quad \nu \tanh \nu = d\,\Omega_1^2/g, \qquad (5.4)$$

$$\mu \tanh \mu = d\Omega_2^2/g.$$

Ω_1 represents the Ω defined in §4 which lies closest to $\sigma/2$ for the $\cos 2\theta$ mode and Ω_2 is the Ω which is closest to σ for the $\cos 2\theta$ mode. The free-surface boundary conditions (2.8) and (2.9) yield

$$\alpha R_1 = i\,\alpha^* (A_2 V_1 + BV_2) \qquad (5.5)$$

$$BR_2 = i\alpha^2 V_3 \qquad (5.6)$$

where

$$R_1 = (d\Omega_1^2/g - d\sigma^2/4g)\cosh \nu \int_0^1 J_1^2(a\nu r/d)r dr \qquad (5.7)$$

$$R_2 = (d\Omega_2^2/g - d\sigma^2/g)\cosh \mu \int_0^1 J_2^2(a\mu r/d)r dr. \qquad (5.8)$$

The functions V_1, V_2 and V_3 are defined in the appendix. The response for the linear forcing A_2 can be calculated from a method described in (Yamamuro 1981). A trivial solution is $\alpha = 0$ and $B = 0$. The questions to be resolved are whether there are any other solutions, and if so, under what conditions.

Eliminating B from equations (5.5) and (5.6) yields

$$\alpha/\alpha^* = i\,(A_2 V_1/V_2 + i\,\alpha^2\, V_2 V_3/(R_1 R_2)) \qquad (5.9)$$

Let

$$\alpha = |\alpha| e^{i\psi}, \quad f = -V_2 V_3/(R_1 R_2) = |f| e^{i\theta_1}$$

and

$$g = i\,A_2 V_2/R_1 = |g| e^{i\theta_2}. \qquad (5.10)$$

Then

$$|\alpha|^2 = [\cos\theta_1 \pm (\cos^2{}_1 - 1 + |g|^2)^{1/2}]/|f| \qquad (5.11)$$

and

$$\psi = \tfrac{1}{2}\left(\theta_2 + \sin^{-1}(|\alpha|^2|f|\sin\theta_1/|g|)\right). \qquad (5.12)$$

Next, B is evaluated via

$$B = i\,\alpha^2 V_3/R_2. \qquad (5.13)$$

In order that there be nontrivial solutions, two conditions must be satisfied. First, $|\alpha|^2$ must be positive and from (5.11), the conditions that must be met are:

(a) $\cos^2\theta_1 - 1 + |g|^2 > 0$,

(b) if $\cos\theta_1 + (\cos^2\theta_1 - 1 + |g|^2)^{1/2} > 0$,

there is at least one nontrivial $|\alpha|$. If

$$\cos\theta_1 - (\cos^2\theta_1 - 1 + |g|^2)^{1/2} > 0 ,$$

then there are two solutions for $|\alpha|$. Secondly, equation (5.12) shows that $|\alpha|^2|f|/|g|$ must be less than or equal to 1 .

If Ω_2 is close enough to σ, then k is approximately μ so that the total velocity potential in the sill region is, approximately,

$$\begin{aligned}\phi \doteq\ & \alpha \cos\theta\, e^{-i\sigma t/2} J_1(a\nu r/d)\cosh \nu(z+1) \\ & + (B+A_2)\cos 2\theta e^{-i\sigma t} J_2(akr/d)\cosh k(z+1) + *.\end{aligned} \qquad (5.14)$$

This approximation may be used for the present example. In this case, computations revealed two ranges of forcing frequencies σ , in which near-resonance occurs. One range lies near $2\Omega_1$ and the other is near Ω_2 . In most of these ranges, the wave amplitudes were calculated to be rather high so that instabilities may occur, after which the present theory may not be applicable. However, at the upper end of the range near $2\Omega_1$, the amplitudes were found to be small enough so that the present steady-state theory might be observable in practice.

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REFERENCES

Abramowitz, M. and Stegun, I. A. (eds.) 1972. Handbook of Mathematical Functions. Washington: Nat. Bur. Stand.

Barnard, B., Pritchard, W. G. and Provis, D. 1981. To appear.

Davis, A. M. and Hood, M. J. 1976. Surface waves normally incident on a submerged horizontal cylinder. Siam. J. Appl. Math. Vol. 31, No. 1, July, 28.

Guza, R. T. & Davis, R. E. 1974. Excitation of edge waves by waves incident on a beach. J. Geophys. Res. Vol. 79, No. 9, 1285-1291.

Havelock, T. H. 1929. Forced surface-waves on water. Phil. Mag., S. 7, Vol. 8, No. 51, Oct., 569-576.

Higher Transcendental Functions. 1953 Bateman Manuscript Project. California Inst. of Technology. A. Erdelyi (ed.). McGraw-Hill.

Longuet-Higgins, M. S. 1967. On the trapping of wave energy round islands. J. Fluid Mech. Vol. 29, part 4, 781-821.

Minzoni, A. A. & Whitham, G. B. 1977. On the excitation of edge waves on beaches. J. Fluid Mech. Vol. 79, Part 2, 273-287.

Phillips, O. M. 1966. The dynamics of the upper ocean. Cambridge University Press.

Rockliff, N. 1978. Finite amplitude effects in free and forced edge waves. Math. Proc. Camb. Phil. Soc., 83, 463.

Yamamuro, Y. 1981. Trapping of water waves above a round sill. MRC Technical Summary Report #2195.

APPENDIX

$$\varepsilon_m = \begin{cases} 1 & \text{if} \quad m = 0 \\ 2 & \text{if} \quad n \neq 0 \end{cases}$$

$$f(k) = \int_{-1}^{0} \cosh^2 k(z+1)dx$$

$$F_m = \frac{\varepsilon_m i^{-m} J_m\left(\frac{a\lambda}{D}\right)}{2\lambda \sinh \lambda}$$

$$F_m^* = \frac{\varepsilon_m i^{-m} J_m'\left(\frac{a\lambda}{D}\right)}{2\lambda \sinh \lambda}$$

$$g(\lambda,k) = \int_{-1}^{0} \cosh\lambda(\delta z+1) \cosh k(z+1)dz$$

$$h(\lambda) = \int_{-\frac{1}{\delta}}^{0} \cosh^2\lambda(\delta z+1)dz$$

$$H_m\left(\frac{a\lambda}{D}\right) = H_m^{(1)}\left(\frac{a\lambda}{D}\right)$$

$$H_m'\left(\frac{a\lambda}{D}\right) = H_m^{(1)'}\left(\frac{a\lambda}{D}\right)$$

$$L(k,z) = J_m(ak/d)\cosh k(z+1) - \left[\frac{H_m^{(1)}(a\lambda/D)\cosh \lambda(\delta z+1)g(\lambda,k)}{H_m^{(1)'}(a\lambda/D)\ \lambda\ h(\lambda)}\right.$$

$$\left. + \sum_{p=1}^{\infty} \frac{K_m(a\lambda_p/D)\ \cos\ \lambda_p(\delta z+1)g(i\lambda_p,k)}{K_m'(a\lambda_p/D)\ \lambda_p\ h(i\lambda_p)}\right](kJ_m'(ak/d)/\delta)$$

$$L_n(k_n,z) = \cos k_n(z+1) - \left[\frac{H_m^{(1)}(a\lambda/D)\cosh \lambda(\delta z+1) g(\lambda,ik_n)}{H_m^{(1)'}(a\lambda/D)\ \lambda\ h(\lambda)}\right.$$

$$\left. + \sum_{p=1}^{\infty} \frac{K_m(a\lambda_p/D)\cos \lambda_p(\delta z+1) g(i\lambda_p,ik_n)}{K_m'(a\lambda_p/D)\lambda_p\ h(i\lambda_p)}\right] \frac{k_n}{\delta} \frac{I_m'(ak_n/d)}{I_m(ak_n/d)}$$

$$X(k,\lambda,\lambda_n) = J_m(ak/d)f(k) - \frac{k\ J_m'(ak/d)}{\delta}\left[\frac{H_m(a\lambda/D)[g(\lambda,k)]^2}{H_m'(a\lambda/D)\ \lambda\ h(\lambda)}\right.$$

$$\left. + \sum_{p=1}^{\infty} \frac{K_m(a\lambda_p/D)[g(i\lambda_p,k)]^2}{K_m'(a\lambda_p/D)\ \lambda_p\ h(i\lambda_p)}\right]$$

$$X_p(k_p,\lambda,\lambda_n) = \frac{-I_m'(ak_p/d)k_p}{I_m(ak_p/d)\ \delta}\left[\frac{H_m(a\lambda/D)g(\lambda,ik_p)g(\lambda,ik_p)}{H_m'(a\lambda/D)\ \lambda\ h(\lambda)}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{K_m(a\lambda_n/D)g(i\lambda_n,k)g(i\lambda_n,ik_p)}{K_m'(a\lambda_n/D)\ \lambda_n\ h(i\lambda_n)}\right]$$

$$Y(k,k_p,\lambda,\lambda_n) = -\frac{k\ J_m'(ak/d)}{\delta}\left[\frac{H_m(a\lambda/D)g(\lambda,ik_p)g(\lambda,k)}{H_m'(a\lambda/D)\ \lambda h(\lambda)}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{K_m(a\lambda_n/D)g(i\lambda_n,ik_p)g(i\lambda_n,k)}{K_m'(a\lambda_n/D)\ \lambda_n h(i\lambda_n)}\right]$$

$$Y(k_p,k_p,\lambda,\lambda_n) = f(ik_p) - \frac{I_m'(ak_p/d)k_p}{I_m(ak_p/d)\delta}\left[\frac{H_m(a\lambda/D)[g(\lambda,ik_p)]^2}{H_m'(a\lambda/D)\ \lambda h(\lambda)}\right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{K_m(a\lambda_n/D)\ [g(i\lambda_n,ik_p)]^2}{K_m'(a\lambda_n/D\ \lambda_n\ h(i\lambda_n)}\right]$$

$$Y(k_s,k_p,\lambda,\lambda_n) = -\frac{k_s}{\delta}\frac{I_m'(ak_s/d)}{I_m(ak_s/d)}\left[\frac{H_m(a\lambda/D)g(\lambda,ik_p)g(\lambda,ik_s)}{H_m'(a\lambda/D)\ \lambda h(\lambda)}\right.$$

$$\left.+\sum_{t=1}^{\infty}\frac{K_m(a\lambda_t/D)g(i\lambda_t,ik_p)g(i\lambda_t,ik_s)}{K_m'(a\lambda_t/D)\ \ \lambda_t\ \ h(i\lambda_t)}\right]$$

$$x_2(k,k_n) = -\frac{kJ_n(ak/d)}{\delta}\sum_{p=1}^{\infty}\frac{K_m(a\lambda_p/D)[g(i\lambda_p,k)]^2}{K_m'(a\lambda_p/D)\ \ \lambda_p\ \ h(i\lambda_p)}$$

$$-\sum_{p=1}^{\infty}\frac{A_{mp}}{A_m}\frac{I_m'(ak_p/d)}{I_m(ak_p/d)}\frac{k_p}{\delta}\left[\frac{H_m(a\lambda/D)\ \ g(\lambda,k)g(\lambda,ik_p)}{H_m'(a\lambda/D)\ \lambda\ h(\lambda)}\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{g(i\lambda_n,k)g(i\lambda_n,ik_p)\ \ K_m(a\lambda_n/D)}{\lambda_n\ \ h(i\lambda_n)\ \ \ \ K_m'(a\lambda_n/D)}\right]$$

$$V_1 = -\tfrac{1}{2}\Big[-2\ \overline{\nu\ \sinh\ \nu}(-\alpha^2 k\ \sinh\ k + k^2\ \cosh\ k) + k\ \sinh\ k\Big(-\frac{\alpha^2}{4}\ \overline{\nu\ \sinh\ \nu}$$

$$+\ \nu\ \sinh\ \nu\Big)\Big] \times \int_0^1 J_1\left(\frac{a\nu r}{d}\right)\ J_2\left(\frac{a\bar{k}r}{d}\right)\ J_1\left(\frac{a\nu r}{d}\right) r\,dr$$

$$+\ \alpha^2\Big(\tfrac{1}{2}k\ \cosh\ k\ \overline{\nu\ \cosh\ \nu}\ \int_0^1 J_1'\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)J_2'\left(\frac{akr}{d}\right)r\,dr$$

$$+\ \frac{d^2}{a^2}\cosh\ k\ \overline{\cosh\ \nu}\ \int_0^1 J_1\left(\frac{a\bar{\nu}r}{d}\right)J_2\left(\frac{akr}{d}\right)J_1\left(\frac{a\nu r}{d}\right)\frac{dr}{r}$$

$$+\ \tfrac{1}{2}k\ \sinh\ k\ \overline{\nu\ \sinh\ \nu}\ \int_0^1 J_1\left(\frac{a\bar{\nu}r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)J_2\left(\frac{akr}{d}\right)r\,dr\Big)$$

where $k\ \tanh\ k = \dfrac{d\sigma^2}{g}$.

V_2 is identical to V_1 but with μ instead of k, where $\mu \tanh \mu = \frac{d\Omega_2^2}{g}$.

$$V_3 = \nu^2 \sinh \nu(-\nu \cosh \nu + \frac{3}{4} \alpha^2 \sinh \nu) \int_0^1 J_2\left(\frac{a\mu r}{d}\right)J_1\left(\frac{a\bar{\nu} r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)rdr$$

$$+ \frac{\alpha^2}{2} \left(\nu^2 \cosh^2\nu \int_0^1 J_2\left(\frac{a\mu r}{d}\right)J_1'\left(\frac{a\bar{\nu} r}{d}\right)J_1'\left(\frac{a\nu r}{d}\right)rdr\right.$$

$$\left.- \frac{d^2}{a^2} \cosh^2\nu \int_0^1 J_2\left(\frac{a\mu r}{d}\right)J_1\left(\frac{a\bar{\nu} r}{d}\right)J_1\left(\frac{a\nu r}{d}\right)\frac{dr}{r}\right)$$

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When the flow over a submerged, round, upright cylinder, situated in a large ocean, is forced by a train of plane waves, linear theory (Yamamuro, 1981) shows that the response can be abnormally large for certain forcing frequencies. The aim of this paper is to present a weakly nonlinear theory,

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where wave interactions, arising from the quadratic terms in the free-surface boundary conditions, can yield abnormally large responses.

A specific interaction will be considered between a flow at a subharmonic frequency and a flow at the driving frequency. The reason for considering such an interaction derived from a consideration of some experimental results of Barnard, Pritchard and Provis (1981).